At the time of issue this report has not been published. Any author desiring to quote this paper should refer to it as "A paper of the Meteorological Research Committee (London), M.R.P. No. 996 by Mr. J. Briggs
A copy is available in the Library of the Meteorological Office."

M.R.P.996
S.C.III/214
20th July, 1956

AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

Condensation Trails formed by Jet Aircraft

by

J. Briggs (Meteorological Office)

1. At present, methods of forecasting the height of likely condensation trail formation are based on the 'Mintra' line and tropopause height. These methods do not give satisfactory results when applied to jet aircraft. In particular it is known that trails frequently continue well into the stratosphere. An attempt is made here to determine forecasting criteria for use with jet aircraft, and persistence of trails is considered.

Symbols used are:-

Velocity of aircraft relative to air	=	V	cm/sec
Rate of fuel consumption	=	M	gm/sec
Amount of heat added to air by combustion of 1 gm of fuel	=	F	calories
Amount of water vapour added to air by combustion of 1 gm of fuel	=	w	gm
Area of cross-section swept out by aircraft	=	A	sq.cm
Relative humidity of air with respect to ice	=	U	
Temperature of air	=	T	^{o}A
Atmospheric pressure	=	p	mb
Saturation water vapour pressure with respect to ice	=	e	mb
Amount of water condensed in air immediately behind aircraft	=	X	gm/sec
Latent heat of sublimation of ice	=	L	cal/gm
Specific heat of air at constant pressure	=	c_p	cal/gm/degree A.
Specific gravity of water vapour with respect to air	=	E	
Density of air	=	ρ	gm/c.c.
Joule constant for mechanical equivalent of heat	=	J = 4.18×10^7	ergs/cal
Gas constant for dry air	=	R = 2.87×10^6	ergs/gm

2. Initial formation of trails

It is assumed that the heat and water vapour supplied by the aircraft engine is mixed thoroughly with the air in the first few yards of wake. It is then possible to accept the area swept out by the aircraft as small; Bannon (1) estimates that A is of the order of 6×10^5 at a distance 20 to 50 yards behind the aircraft.

Using the First Law of Thermodynamics we have

$$FM + LX = c_p . A \rho V . \delta T \qquad 2.1$$

/Also,

Also, since water vapour is not destroyed,

$$\omega M + \frac{Ee}{p}\cdot\frac{U}{100}\cdot A\rho V = X + \frac{E(e+\delta e)}{p}\cdot A\rho V$$

which, using the Clausius Clapeyron equation $\frac{de}{dT} = \frac{JeEL}{RT^2}$, becomes

$$\omega M + \frac{Ee}{p}\cdot\frac{U}{100}\cdot A\rho V = X + \frac{Ee}{p}\left(1+\frac{JEL\,\delta T}{RT^2}\right)\cdot A\rho V \qquad 2.2.$$

Eliminating δT between 2.1. and 2.2. we have

$$X = \frac{\omega M - \frac{Ee}{p}\left(1-\frac{U}{100}\right)A\rho V - \frac{E^2eJLFM}{Rpc_pT^2}}{1+\frac{E^2eJL^2}{Rpc_pT^2}} \qquad 2.3.$$

Now if the air is near saturation then $1-U/100$ approaches zero and it is possible to ignore the middle term in the numerator since A is also relatively small near the aircraft and we can write

$$X = \frac{\omega M - \frac{E^2eJLFM}{Rpc_pT^2}}{1+\frac{E^2eJL^2}{Rpc_pT^2}} \qquad 2.4.$$

Saturation is reached when X=0 but trails are unlikely to become visible until X exceeds a certain critical value. From 2.4. it can be shown that

$$\frac{dX}{dT} = \frac{E^2eJLpc_pM(F+\omega L)(2TR-EJL)}{(Rpc_pT^2+E^2eJL^2)^2}$$

and since, for all normal atmospheric temperatures, $2TR \ll EJL$ then dX/dT is negative.

Putting $X = 0$, we get as the criterion for trails to be just possible in a near-saturated atmosphere

$$T = \left(\frac{E^2eJLF}{Rpc_p\omega}\right)^{1/2} \qquad 2.5.$$

If T exceeds this value then X is negative.

This critical value is independent of M, the rate of fuel consumption, and is determinate if F and ω are known.

For jet aircraft, using paraffin, we have $\omega = 1.3$ gm and $F = 10^4$ calories (assuming all the heat of the combustion is available to heat the air just behind the aircraft).

/If

If saturation with respect to ice is the critical state then the trail forms as ice particles. It is possible that trail formation may be delayed until saturation with respect to water is reached but to determine the absolute threshold of trail formation it is necessary to use the value of L for ice, i.e., $L = 677$ cal/gm.

2.5. then reduces to $$T = 11060\left(\frac{e}{p}\right)^{1/2} \qquad 2.6.$$

Brunt (Physical and Dynamical Meteorology-p.405) gives data showing the variation of e. Using these data 2.6. can be solved for T, given p, and yields:-

Table I

Variation of critical temperature, for initial trail formation, with pressure

Pressure (mb)	Corresponding critical temperature (°C)
150	-45.4
200	-43.0
250	-41.1
300	-39.5
350	-38.0
400	-36.6
450	-35.3
500	-34.0

The figures of Table I correspond to the Mintra values for piston-type aircraft and serve as a criterion of the possible initial formation of condensation trails. If the air is not near saturation then it is reasonable to suppose that the critical temperature will be somewhat lower than that shown by Table I. Bannon (3) suggests that the humidity has little effect on the initial formation of trails.

3. Persistency of trails

Frost (4) has suggested that the diffusion of particles (or droplets) formed immediately behind the aircraft can be treated as a continuous point source problem with the indicated air speed of the aircraft replacing the mean wind of the point source problem. It is unlikely that the actual diffusion will correspond with that in conditions in which the mean wind is as high as the speed of a jet aircraft.

It seems more reasonable to assume that the mixing is complete at a very short distance behind the aircraft and that thereafter any particles or droplets formed are diffused normally. In view of the high speed of jet aircraft it is suggested that the problem should be treated as one of diffusion of droplets produced almost instantaneously along a line (the line of flight of the aircraft).

Sutton (5), considering diffusion from an instantaneous point source, has obtained the expression

$$\chi(x,y,z,t) = \frac{\chi_0}{\pi^{3/2} C_x C_y C_z (\bar{u}t)^{\frac{3}{2}m}} \exp\left[-(\bar{u}t)^{-m}\left(\frac{x^2}{C_x^2} + \frac{y^2}{C_y^2} + \frac{z^2}{C_z^2}\right)\right] \qquad 3.1.$$

/where

where χ is the amount of matter instantaneously produced at time $t = 0$ and $\chi(x,y,z,t)$ represents the concentration at a point (x,y,z) at time t. (x,y,z) is measured from an origin moving with the cloud at constant speed $\bar{u}$ (the mean wind speed). C_x, C_y, C_z are coefficients of diffusion in directions parallel to the co-ordinate axes. Sutton shows that $m = 7/4$ gives reasonably accurate results over distances varying from metres to hundreds of kilometres. Using 3.1. and taking axes moving with the mean wind speed and the x axis parallel to the line of flight of the aircraft we have:-

amount of condensed water vapour at (x,y,z) at time t due to original condensed water vapour at $(x',0,0)$ is

$$\frac{(X/V)\,dx'}{\pi^{\frac{3}{2}} C_x C_y C_z (\bar{u}t)^{\frac{3}{2}m}} \exp\left[-(\bar{u}t)^{-m}\left(\frac{(x-x')^2}{C_x^2} + \frac{y^2}{C_y^2} + \frac{z^2}{C_z^2}\right)\right] \qquad 3.2.$$

since the amount of condensed water vapour in unit length of the original trail is X/V.

The line source is assumed infinite and then from 3.2. the total amount of condensed water vapour at (x,y,z) and at time t is

$$\frac{X/V}{\pi^{\frac{3}{2}} C_x C_y C_z (\bar{u}t)^{\frac{3}{2}m}} \int_{-\infty}^{+\infty} \exp\left[-\frac{\frac{(x-x')^2}{C_x^2} + \frac{y^2}{C_y^2} + \frac{z^2}{C_z^2}}{(\bar{u}t)^m}\right] dx'$$

$$= \frac{X/V}{\pi C_y C_z (\bar{u}t)^m} \exp -\left\{\frac{\frac{y^2}{C_y^2} + \frac{z^2}{C_z^2}}{(\bar{u}t)^m}\right\} \qquad 3.3.$$

If the particles originally formed are spherical and of uniform radius r_0 we have

$$X = \tfrac{4}{3}\pi r_0^3 \rho_i N \qquad \text{where } \rho_i = \text{density of ice}$$

N = no. of particles formed per second and using 3.3. the number of particles at (x,y,z) at time t is

$$\frac{N/V}{\pi C_y C_z (\bar{u}t)^m} \exp -\left\{\frac{\frac{y^2}{C_y^2} + \frac{z^2}{C_z^2}}{(\bar{u}t)^m}\right\} \qquad 3.4.$$

Due to evaporation the particle will now have a radius r_t where r_t is connected with r_0 by the relation (cf. Frost (4))

$$r_t^2 = r_0^2 - \frac{2Dt}{\rho_i}(\rho_s - \rho_a) \qquad 3.5.$$

in which D = coefficient of diffusion of water vapour in air
ρ_s = vapour density at surface of particle
ρ_a = vapour density in free air.

/From

From 3.4. the number of particles in a section of the trail at a fixed distance z from the axis at time t is

$$\frac{N/V}{\pi C_y^2 C_z^2 (\bar{u}t)^m} \exp\left\{-z^2/(C_z^2(\bar{u}t)^m\right\} \int_{-\infty}^{+\infty} \exp\left\{-y^2/C_y^2(\bar{u}t)^m\right\} dy$$

$$= \frac{N/V}{\pi^{\frac{1}{2}} C_z (\bar{u}t)^{m/2}} \exp\left\{-x^2/C_z^2(\bar{u}t)^m\right\} \qquad 3.6.$$

The trail is visible when the illumination from the particles is sufficient to obscure the background. The scattering produced by any single particle is proportional to the cross-section of the drop and hence proportional to the square of the radius. Therefore the scattering of incident sunlight by all the particles in the section (of 3.6.) is

$$\frac{k(N/V) r_t^2}{\pi^{\frac{1}{2}} C_z (\bar{u}t)^{m/2}} \exp\left\{-z^2/C_z(\bar{u}t)^m\right\} \qquad 3.7.$$

in which k is a constant.

If the amount of scattered light required to obscure the background of the sky is P then from 3.7. the radius of the visible trail is given by

$$P = \frac{k(N/V) r_t^2}{\pi^{\frac{1}{2}} C_z (\bar{u}t)^{m/2}} \exp\left\{-R^2/C_z^2(\bar{u}t)^m\right\} \qquad 3.8.$$

Figure 1 shows a cross-section of the trail. The shaded circle represents the visible part of the trail whilst the outer circle represents the whole spread of particles. The observer sees only the shaded portion since in any cross-section for which z exceeds R, the visible radius, the light scattered by the particles in the cross-section is not sufficient to obscure the background.

The trail vanishes when the visible radius $R = 0$ or, from 3.8., when

$$P = \frac{k(N/V) r_{t_c}^2}{\pi^{\frac{1}{2}} C_z (\bar{u}t_c)^{m/2}} \qquad 3.9.$$

where t_c is the time during which the trail persists.

If L = length of visible trail we may write, taking axes with the origin moving with the aircraft, $x = Vt$ and $L = Vt_c$.

Elimination of P between 3.8.and 3.9.gives

$$\frac{r_t^2}{(t)^{m/2}} \exp\left\{-R^2/C_z^2(\bar{u}t)^m\right\} = \frac{r_{t_c}^2}{(t_c)^{m/2}}$$

/or

or neglecting evaporation so that $r_t = r_{t_c} = r_0$ we have

$$\frac{\exp\left\{-R^2/C_z^2(\bar{u}x/V)^m\right\}}{(\bar{u}x/V)^{m/2}} = \frac{1}{(\bar{u}L/V)^{m/2}}$$

or $$\exp\left\{R^2/C_z^2(\bar{u}x/V)^m\right. = (L/x)^{m/2}$$

i.e., $$R^2 = C_z^2\left(\frac{\bar{u}}{V}\right)^m \cdot \frac{m}{2} x^m \log_e \frac{L}{x}$$

and putting y for R, the equation of the outline of the visible trail can be written in the form

$$y^2 = A x^m \log_e L/x \qquad 3.10.$$

where $$A = C_z^2\left(\frac{\bar{u}}{V}\right)^m \cdot \frac{m}{2}$$

This equation agrees with that given by Frost (4) in his equation 3.08. if $\bar{u} = V$ as Frost, in effect, assumes.

Figure 2 illustrates successive cross-sections of the trail and shows the continuous spread of particles together with a growth then diminution of the visible trail. The outline of the trail which is obtained by joining the successive shaded portions of Figure 2 can be compared with photographs of trails (cf. Figure 1 in Frost's paper (4)).

Returning to equation 3.9. and substituting for N and r_t we have

$$\frac{X}{\frac{4}{3}\pi r_0^3 \rho_i} = \frac{P}{k} \cdot \frac{\pi^{\frac{1}{2}} C_z (\bar{u}t_c)^{m/2} V}{r_0^2 - \frac{2Dt_c(\rho_s - \rho_a)}{\rho_i}}$$

so that the condition for the trail to remain visible for a minimum time t_c is given by

$$X = \frac{P}{k} \cdot \frac{\frac{4}{3}\pi^{3/2} \rho_i r_0^3 C_z (\bar{u}t_c)^{m/2} V}{r_0^2 - \frac{2Dt_c(\rho_s - \rho_a)}{\rho_i}} \qquad 3.11.$$

or

$$X = \frac{K_1 \bar{u}^{m/2} r_0 V}{1 - \frac{K_2(\rho_s - \rho_a)}{r_0^2}} \qquad 3.12.$$

where $K_1 = \frac{4}{3}\frac{P}{k}\pi^{3/2}\rho_i t_c^{m/2} C_z$ and is approximately constant,

and $K_2 = 2Dt_c/\rho_i$.

For near saturated air $(\rho_s - \rho_a)$ approximates to zero and it is then possible to combine 2.4. and 3.12. to give

$$wM - \frac{E^2 e J L F M}{R\rho c_p T^2} = \left\{1 + \frac{E^2 e J L^2}{R\rho c_p T^2}\right\} K_1 w^{m/2} r_0 V \qquad 3.13.$$

This equation indicates the conditions under which a visible trail will persist for a minimum time t_c in near-saturated air.

Trail persistency depends therefore on a number of factors (including air temperature, humidity, rate of fuel consumption, indicated airspeed, mean wind speed, and radius of particles condensed immediately behind aircraft). It is possible to say that, for any particular aircraft, persistent trails are more likely the higher the humidity, the lower the temperature and the lower the mean wind.

The complexity of equation 3.13. suggests that simple forecasting criteria can hardly be expected to indicate more than broad average conditions. By assuming reasonable values it may however be possible to check the equation.

The equation can be rewritten in the form

$$\frac{T^2}{e} = \frac{E^2 J L F}{R\rho c_p w} \cdot \frac{1 + K_1 w^{m/2} r_0 L (V/M)(1/F)}{1 - K_1 w^{m/2} r_0 (V/M)(1/w)}$$

$$= \frac{T_i^2}{e_i} \cdot \frac{1 + K_1 w^{m/2} r_0 L (V/M)(1/F)}{1 - K_1 w^{m/2} r_0 (V/M)(1/w)} \qquad 3.14.$$

where T_i, e_i are values corresponding to initial trail formation and satisfy equation 2.5.

Experimental values suggest r_0 of the order of 15 μ (cf. Mason (6)) whilst Frost, considering a trail against the blue sky with the sun in the zenith, reaches a value which is equivalent to $P/k = 8.5 \times 10^{-4}$. Also, from values reached by various experimenters, Frost takes $C_z = 0.30$ cm and $m = 7/4$.

Since $t_c = 10$ sec corresponds to a trail length of one mile or more it is considered that this is a reasonable value to take.
We have then

$$K_1 = \frac{4}{3}\frac{P}{k}\pi^{3/2}\rho_i t_c^{m/2} C_z$$

$$= 1.3 \times 10^{-2} \quad \text{approx.}$$

whilst performance figures for Sabre aircraft suggest V/M is approximately constant at 50 cm/gm.

/In

In equation 3.14. it is necessary that

$$1 > K_1 u^{m/2} r_0 (v/M)(1/w) \quad \text{since} \quad T^2/e > 0$$

Hence $$K_1 u^{m/2} r_0 (v/M)(L/F) \ll 1 \quad \text{since} \quad wL/F \ll 1$$

and therefore

$$\frac{T^2}{e} \simeq \frac{T_i^2}{e_i} \frac{1}{1 - K_1 u^{m/2} r_0 \frac{V}{M} \frac{1}{w}} \qquad 3.15.$$

or, with the constants assumed, we have that trails can only persist for 10 seconds if

$$u^{7/8} < 1333$$

i.e., $u < 3726$ cm/sec

or $u < 75$ knots (approx).

The critical temperature is now given by

$$\frac{T^2}{e} = \frac{T_i^2}{e_i} \frac{1}{1 - 7.5 \times 10^{-4} u^{7/8}}$$

Substituting for u we obtain Table II.

Table II

Theoretical critical temperatures required for visible trails to persist at least 10 seconds under varying winds and pressures and in near-saturated air. (Figures in degrees C)

		Wind in knots			
		60	50	40	30
Pressure in mb	150	-58	-55	-53	-51
	200	-56	-53	-51	-49
	250	-55	-52	-49	-47
	300	-54	-51	-47	-45
	350	-53	-50	-46	-44
	400	-52.5	-48	-45	-43
	450	-52	-47	-44	-42
	500	-51	-46	-43	-40.5

4. Comparison with actual results

Some 94 aircraft reports of condensation trails were forwarded by outstations to Watnall during the months of October, November and December, 1954. In these reports were given the aircrew estimates of the height of formation of trails, both persistent and non-persistent, and the meteorological officer's estimates of temperatures at the heights, places and times concerned.

These estimates of temperatures at the base, or tops, of reported persistent trails were regarded as being the critical temperatures for formation of persistent trails under the conditions prevailing. A total of 96 values was obtained and was expressed in a contingency table as follows.

/Table III

Table III

Reported height for base (or tops) of a persistent trail and corresponding critical temperature (estimated from relevant tephigrams)

Temperature (°A)

Height (feet)	207	210	213	216	219	222	225	228	231	234	237	240	Total
20000													
22000									½				½
24000							1	0	1½	½			3
26000					1	1	1	0	2	0			5
28000					1	2¼	1¾	2½	4	½	1½		13½
30000					1½	2½	3	1½	1				9½
32000			¼	1½	2½	2½	2¾	1½	½				11½
34000			¼	3	4¾	4	2						14
36000		1	½	2½	2¼	3¼							9½
38000	¼	1¾	1	1½	1								5½
40000	1½	4¼	2¾	1½	1½	½							12
42000	1¼	3½	2¼	2½	1								10½
44000		½	0	½	½								1½
Total	3	11	7	13	17	16	11½	5½	9½	1	1½		96

The table yields a correlation between the height H and the critical temperature of -0.8. The corresponding lines of regression are:-

$$T = 256.7 - 0.001052\,H \quad \ldots \quad \ldots \quad 4.1.$$
$$H = 171500 - 623.7\,T \quad \ldots \quad \ldots \quad 4.2.$$

H in feet, T in degrees Absolute.

No attempt was made to alter aircrew estimates of persistency in the above analysis except that two cases reported as semi-persistent were clsssified as persistent. The good correlation between T and H therefore indicates that temperature alone could be used as a good guide to forecasting the height at which aircrew would begin to estimate trail formation as persistent. Equation 4.1. yields the following figures for critical temperature:-

at 150 mb	-63°C	at 350 mb	-44°C	4.3.
200 mb	-57°C	400 mb	-41°C	
250 mb	-52°C	450 mb	-38°C	
300 mb	-48°C	500 mb	-36°C	

The theoretical values shown in Table II above are in reasonable agreement with 4.3. and indeed if comparison is limited to the range 250 to 350 mb, i.e., the zone in which more than two-thirds of the reported cases occurred, then 4.3. agrees very closely indeed with the values corresponding to a mean wind of 40 knots. No actual wind observations were available but 40 knots is a reasonable mean value for winds over England between 250 and 350 mb in autumn. (The mean wind at 300 mb over Larkhill in November was 37 knots for the years 1946-50.)

5. Conclusions

It is suggested that aircrew may usefully be told:-

a. the height above which trails are likely
b. the zone in which trails are likely to persist.

/(a)

(a) could be based on Table I which would replace the existing Mintra values. Section 3 shows the complicated problem involved in (b) but Section 4 suggests that empirical values on the lines of equation 4.1. could be used: the values probably differing slightly for each type of aircraft. Alternatively (b) could be based on a combination of wind and temperature, provided the air is near saturation, by using the figures of Table II. Similar tables could be constructed for different persistence times.

References

1. J.K. Bannon - Meteorological Research Papers, Air Ministry, London, No. 489.
2. W.L. Godson - Paper issued by Met. Division, Dept. of Transport, Canada.
3. J.K. Bannon - Meteorological Research Papers, Air Ministry, London, No. 476.
4. R. Frost - Meteorological Research Papers, Air Ministry, London, No. 830.
5. O.G. Sutton - Micrometeorology.
6. B.J. Mason - Q.J.R.M.S., 79, p.104.

FIGURE 1

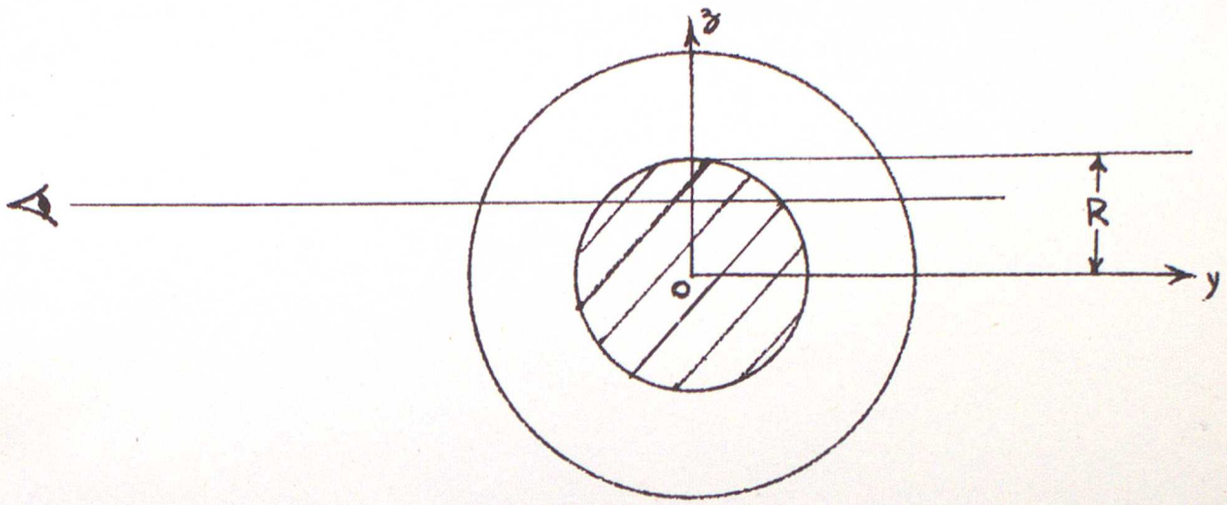
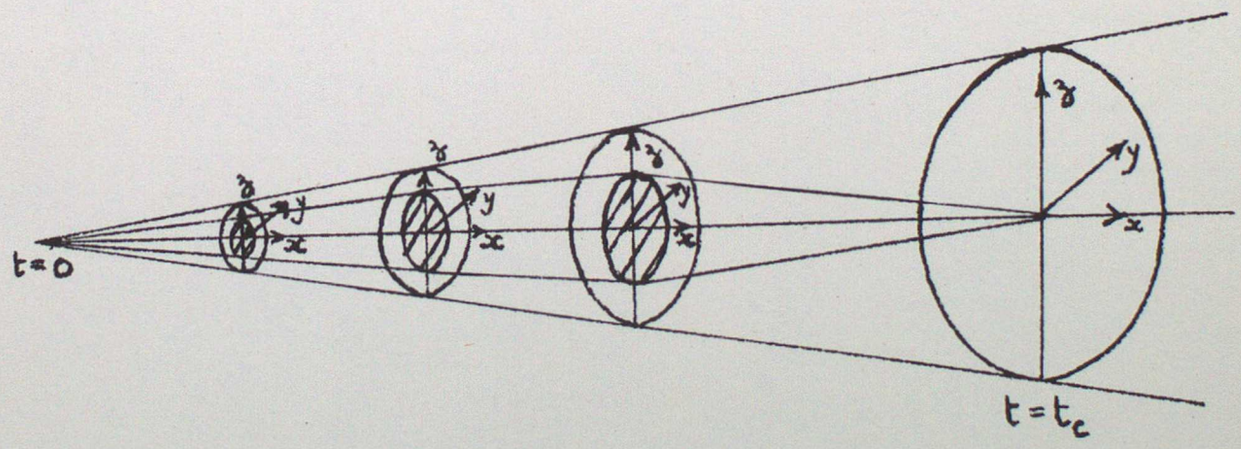


FIGURE 2



CONDENSATION TRAILS FORMED BY JET AIRCRAFT